VARIATIONAL CHARACTERIZATIONS OF THE TOTAL SCALAR CURVATURE AND EIGENVALUES OF THE LAPLACIAN

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ABSTRACT. For the dual operator $s_g'^*$ of the linearization s_g' of the scalar curvature function, it is well-known that if $\ker s_g'^* \neq 0$, then s_g is a non-negative constant. In particular, if the Ricci curvature is not flat, then $s_g/(n-1)$ is an eigenvalue of the Laplacian of the metric g. In this work, some variational characterizations were performed for the space $\ker s_g'^*$. To accomplish this task, we introduce a fourth-order elliptic differential operator $\mathcal{A}$ and a related geometric invariant ν. We prove that ν vanishes if and only if $\ker s_g'^* \neq 0$, and if the first eigenvalue of the Laplace operator is large compared to its scalar curvature, then ν is positive and $\ker s_g'^* = 0$. Furthermore, we calculated the lower bound on ν in the case of $\ker s_g'^* = 0$. We also show that if there exists a function which is $\mathcal{A}$-superharmonic and the Ricci curvature has a lower bound, then the first non-zero eigenvalue of the Laplace operator has an upper bound.

1. INTRODUCTION

Let M be a compact smooth n-manifold (without a boundary). The space of all Riemannian metrics, $\mathcal{M}$, on M is then open in the space of symmetric 2-tensors, $\mathcal{S}^2(M)$, for the compact-open topology or the $W^{k,p}$-topology, where $W^{k,p}$ denotes the Sobolev space. For a Riemannian metric g and a symmetric 2-tensor h, the differential of the scalar curvature at g in the direction h, $s_g'(h)$, is given by

$$(1.1) \qquad s_g'(h) = -\Delta_g \mathrm{tr}(h) + \delta_g(\delta_g h) - g(r_g, h),$$

where Δ_g is the negative Laplacian of g, and r_g and δ_g denote the Ricci curvature and divergence operator of g, respectively ([3]). In addition, the L^2-adjoint operator

$s_g'^*$ of s_g' is given by

(1.2)
$$s_g'^*(f) = Ddf - (\Delta_g f)g - fr_g,$$

where Ddf denotes the Hessian of f with respect to the metric g. Note that both s_g' and $s_g'^*$ are linear second order differential operators.

In this paper, we consider a fourth-order elliptic differential operator $\mathcal{A} = s_g' \circ s_g'^* : C^\infty(M) \to C^\infty(M)$. The existence of homogeneous or non-homogeneous solutions to $\mathcal{A}$ is deeply related to the kernel space of $s_g'^*$. For example, in [4] and [5], Bourguignon, and Fischer-Marsden proved, respectively, that if $\ker s_g'^* \neq 0$, then either (M, g) is Ricci-flat and $\ker s_g'^* = \mathbb{R} \cdot 1$, or the scalar curvature is a strictly positive constant and $\frac{s_g}{n-1}$ is an eigenvalue of the Laplacian. On the other hand, if $\ker s_g'^* = 0$, then for any function $\eta \in C^\infty(M)$ there exists a unique function $u \in C^\infty(M)$ such that $\mathcal{A}(u) = \eta$ (Theorem 2.2). In fact, the condition $\ker s_g'^* = 0$ implies the injectivity of $s_g'^*$ and the surjectivity of s_g'. In order to perform variational characterizations of the condition $\ker s_g'^* \neq 0$, we introduce a geometric invariant ν which is defined by

$$\nu = \inf \left\{ \int_M \varphi \mathcal{A} \varphi \, dv_g \right\},$$

where the infimum is taken over all functions $\varphi \in H^2(M) = W^{2,2}(M)$ with $\int_M \varphi^2 = 1$. Here $H^2(M) = W^{2,2}(M)$ denotes the Sobolev space which is L^2 up to the second (weak) derivatives.

A basic result related to the invariant ν is the following.

Theorem A $\nu > 0$ if and only if $\ker s_g'^* = 0$.

For the case $\ker s_g'^* = 0$, we give a lower bound on ν and its relationship with the first non-zero eigenvalue of the Laplacian. We also show that if the first eigenvalue is large compared to the scalar curvature, then ν is positive and $\ker s_g'^* = 0$. In addition, if M is the product of two standard spheres of the same dimension, then ν is exactly equal to the dimension of the sphere.

Theorem B Let $M = S^m \times S^m$ ($m \geq 2$) with the standard product metric. Then
$$\nu = m = \frac{\dim(M)}{2}.$$

We also obtain upper bounds for the first non-zero eigenvalue of the Laplace operator when $\mathcal{A}$ satisfies a condition on evaluating functions. We say that a Riemannian manifold (M, g) satisfies the $\mathcal{A}$-superharmonic condition if there exists a smooth function φ such that $M_\varphi^+ \neq \emptyset$ and $\mathcal{A}\varphi \leq 0$ on M_φ^+, and $\Delta\varphi = 0$ on the boundary ∂M_φ^+ of M_φ^+. For example, if M is the standard sphere, then the first eigenfunction of the Laplacian satisfies those conditions. In general, any compact Riemannian manifold (M, g) with a positive scalar curvature and $\ker s_g'^* \neq 0$ satisfies the $\mathcal{A}$-superharmonic condition.

One of our main results is the following.

Theorem C Let (M^n, g) be a compact n-dimensional Riemannian manifold with a positive constant scalar curvature s_g. Suppose that (M, g) satisfies the $\mathcal{A}$-superharmonic condition. If $r_g \geq k$ for $0 < k \leq 2\left(1 - \sqrt{1 - \frac{1}{n}}\right) s_g$, then the first eigenvalue of the Laplacian satisfies

$$(1.3) \qquad \lambda = \lambda_1(M) \leq \frac{2s_g - k + \sqrt{k^2 - 4ks_g + \frac{4s_g^2}{n}}}{2(n-1)}.$$

The inequality (1.3) is sharp since the equality holds for the standard sphere. In performing analysis with the operator $\mathcal{A}$, main difficulty is that we cannot apply the theory of second order elliptic PDE directly since $\mathcal{A}$ is a fourth-order differential operator. In fact, there are not much known results for fourth-order ellitptic PDE.

The kernel space of $s_g'^*$ plays an important role in the critical point equation arising from the total scalar curvature functional. Let $\mathcal{M}_1$ be the set of all smooth Riemannian metrics of unit volume on M, and let $\mathcal{C} \subset \mathcal{M}_1$ be the set of all smooth Riemannian metrics on M with a constant scalar curvature, i.e.,

$$\mathcal{C} = \{g \in \mathcal{M} : \mathrm{vol}(M, g) = 1 \text{ and } s_g = \text{constant}\}.$$

The total scalar curvature $\mathcal{S} : \mathcal{M}_1 \to \mathbb{R}$ is defined as

$$\mathcal{S}(g) = \int_M s_g \, dv_g.$$

It is well-known that the total scalar curvature functional $\mathcal{S}$ restricted to $\mathcal{C}$ will be critical at g if and only if there is a function f with $\int_M f = 0$ such that

(1.4) $$z_g = s_g'^*(f)$$

where z_g is the trace-less Ricci tensor defined as $z_g = r_g - \frac{s_g}{n}g$. We call (1.4) the critical point equation (CPE hereafter). Note that if $f = 0$, then it follows from (1.4) that $z_g = 0$ and thus, g is an Einstein metric. However, the existence of a nonzero solution is a very strong condition. The only known case with a nonzero solution is that of a standard sphere, and it has been conjectured that this is the only possible case ([3]). Namely, it is believed that if there exists a non-zero function f satisfying the CPE, then g must be an Einstein metric. We would like to remark that a solution (g, f) to CPE is a non-trivial example of $\mathcal{A}$-superharmonic condition since $Af = -|z_g|^2$ and $\Delta_g f = -\frac{s_g}{n-1}f$.

Unless stated otherwise, we only consider Riemannian metrics on M whose scalar curvatures are positive constants.

2. Variational Properties

Let (M, g) be a closed Riemannian n-manifold and δ be the adjoint operator of the differential d with respect to the metric g. Unless explicitly stated, we will use r rather than r_g as the Ricci tensor of the metric g, and s rather than s_g as the scalar curvature. The following expressions are well-known definitions and identities: for a function φ and any tensor T,

$$\delta Dd\varphi = -d\Delta\varphi - r(d\varphi, \cdot), \quad \delta d\varphi = -\Delta\varphi$$

and

$$\delta(\varphi T) = \varphi\delta T - T(d\varphi, \cdot).$$

Lemma 2.1. *Let $\mathcal{A} = s_g' \circ s_g'^*$ and assume the scalar curvature $s_g = s$ is constant. Then for any function φ,*

$$\mathcal{A}(\varphi) = (n-1)\Delta^2\varphi + 2s\Delta\varphi - \langle Dd\varphi, r\rangle + \varphi|r|^2.$$

Proof. It follows directly from (1.2) that

$$s_g'^*(\varphi) = Dd\varphi - (\Delta\varphi)g - \varphi r$$

and thus

$$A(\varphi) = s'_g \circ s'^*_g(\varphi) = s'_g(Dd\varphi - (\Delta\varphi)g - \varphi r).$$

By (1.1), we have

$$s'_g(Dd\varphi) = -\delta(r(d\varphi, \cdot)) - \langle Dd\varphi, r \rangle.$$

Similarly, since $\delta g = 0$ and $\delta r = -\frac{1}{2}ds = 0$, we also obtain the following from (1.1)

$$s'_g((\Delta\varphi)g) = (1-n)\Delta^2\varphi - s\Delta\varphi$$

and

$$s'_g(\varphi r) = -s\Delta\varphi + \delta(-r(d\varphi, \cdot)) - \varphi|r|^2.$$

Combining these two expressions, we obtain

$$\mathcal{A}(\varphi) = (n-1)\Delta^2\varphi + 2s\Delta\varphi - \langle Dd\varphi, r \rangle + \varphi|r|^2.$$

$\square$

Note that $\mathcal{A}$ is a fourth-order linear partial differential operator. The following theorem shows that $\mathcal{A}$ is elliptic and self-adjoint. We say that a fourth-order differential operator is elliptic if the symbol is injective.

Theorem 2.2. *The operator $\mathcal{A}$ is a self-adjoint, fourth-order elliptic linear operator. Furthermore, if $\ker s'^*_g = 0$, then for any $\psi \in C^\infty(M)$ there exists a unique function $u \in C^\infty(M)$ such that $\psi = \mathcal{A}(u)$.*

Proof. We first show that s'^*_g has injective symbol. Recall that for any $p \in M$ and any cotangent vector $t \in T^*_p M$, there is a linear map $\sigma_t(s'^*_g) : T_p C^\infty(M) \to T_p C^\infty(S^2 M)$ called the symbol of $D = s'^*_g$, and the symbol of D is called injective if $\sigma_t(D)$ is injective for all non-zero t. Note that for $t \in T^*M$, $\psi \in C^\infty(M)$,

$$\sigma_t(s'^*_g) \cdot \psi = (-g(t,t)g + t \otimes t)\psi,$$

which is clearly injective for $n > 1$. Thus s'^*_g is an operator of order 2 with injective symbol. By Lemma 4.4 of [1], $\mathcal{A} = s'_g \circ s'^*_g$ is an elliptic operator of order 4.

To see the self-adjoint property of $\mathcal{A}$, first of all, note that for any two functions φ, ψ

(2.1) $$\psi\langle Dd\varphi, r \rangle = \mathrm{div}(\psi r(d\varphi, \cdot)) - r(d\varphi, d\psi)$$

and so

$$\int_M \psi \langle Dd\varphi, r \rangle = - \int_M r(d\varphi, d\psi).$$

Integrating $\varphi \mathcal{A} \psi$, we obtain

$$\int_M \psi \mathcal{A}(\varphi) = \int_M (n-1)\Delta\varphi\Delta\psi - 2s_g \langle d\varphi, d\psi \rangle + r(d\varphi, d\psi) + |r|^2 \varphi\psi.$$

Thus,

$$\int_M \varphi \mathcal{A} \psi = \int_M \psi \mathcal{A} \varphi.$$

Secondly, we show that $\mathcal{A}$ is surjective. Since s'_g is surjective, for any non-trivial $\psi \in C^\infty(M)$, there is $\xi \in C^\infty(S^2 M)$ such that $s'_g(\xi) = \psi$. From the fact that s_g is constant and the proof of Theorem 5.2 in [1], $C^\infty(S^2 M) = \operatorname{im} s'^*_g \oplus \ker s'_g$. Thus, $\xi = \xi_1 + \xi_2$ for $\xi_1 \in \operatorname{im} s'^*_g$ and $\xi_2 \in \ker s'_g$. Therefore, for $\xi_1 = s'^*_g(u)$, we have $\mathcal{A}(u) = \psi$.

Finally uniqueness comes from the assumption that $\ker s'^*_g = 0$ since $\ker \mathcal{A} = \ker s'^*_g$; clearly $\ker s'^*_g \subset \ker s'_g \circ s'^*_g$, and $s_g \circ s'^*_g(u) = 0$ implies

$$0 = (u, s_g \circ s'^*_g(u))_{L^2} = (s'^*_g(u), s'^*_g(u))_{L^2},$$

where $(f,g)_{L^2} = \int_M fg \, dv_g$, and so $s'^*_g(u) = 0$. $\qquad\square$

Given a smooth compact n-dimensional Riemannian manifold (M, g), we let $H^2(M) = W^{2,2}(M)$ be the Sobolev space defined as the completion of the space of smooth functions on M with respect to the norm

$$\|\varphi\|^2_{H^2(M)} = \int_M |Dd\varphi|^2 \, dv_g + \int_M |\nabla\varphi|^2 \, dv_g + \int_M \varphi^2 \, dv_g.$$

To investigate the properties of operator $\mathcal{A}$ from the perspective of the calculus of variations, we define $E(\varphi)$ for any function $\varphi \in H^2(M)$ as

$$(2.2) \qquad E(\varphi) = \frac{1}{2} \int_M \left[(n-1)(\Delta\varphi)^2 - 2s|d\varphi|^2 + r(d\varphi, d\varphi) + \varphi^2 |r|^2 \right].$$

According to (2.1) we have $\varphi\langle Dd\varphi, r \rangle = \operatorname{div}(\varphi r(d\varphi, \cdot)) - r(d\varphi, d\varphi)$ and thus

$$\int_M \varphi \langle Dd\varphi, r \rangle = - \int_M r(d\varphi, d\varphi).$$

Consequently, the Euler-Lagrange equation for the functional E is exactly

$$\mathcal{A}(\varphi) = (n-1)\Delta^2\varphi + 2s\Delta\varphi - \langle Dd\varphi, r \rangle + \varphi|r|^2.$$

Note that if $\varphi = $ constant and $\mathcal{A}(\varphi) = 0$, then $\varphi = 0$ if the Ricci curvature r does not identically vanish. Furthermore,

$$(2.3) \qquad E(\varphi) = \frac{1}{2}\int_M \varphi\mathcal{A}(\varphi) = \frac{1}{2}\int_M |s_g'^* \varphi|^2 \geq 0$$

for any function φ.

A simple direct observation is as follows

Lemma 2.3. $\ker s_g'^* = 0$ *if and only if* $\ker \mathcal{A} = 0$.

Proof. The proof follows from the fact that

$$\int_M (s_g'^* \varphi)^2 = \int_M \varphi\mathcal{A}(\varphi)$$

for any function φ. In fact, assume that $\ker \mathcal{A} = 0$ and let $s_g'^* u = 0$. Then u realizes the infimum of $E(\varphi)$ among all smooth functions $C^\infty(M)$. That is, u is a critical point for E and thus, $\mathcal{A}(u) = 0$. $\qquad\square$

Example 2.4. Let M be a round n-sphere S^n with a standard round metric. Also, let φ be the first eigenfunction for the Laplacian so that

$$\Delta\varphi = -n\varphi, \quad \int_{S^n} |d\varphi|^2 = n\int_{S^n}\varphi^2.$$

Since $r = (n-1)g$, it is easy to see that $E(\varphi) = 0$. Thus the first eigenfunction φ realizes the infimum of the functional E and so

$$\mathcal{A}(\varphi) = 0 \quad \text{and} \quad \ker s_g'^* \neq 0.$$

On the other hand, consider $M = S^2 \times S^3$ with the product metric. It can be seen that the scalar curvature s, the first non-zero eigenvalue λ_1 of the Laplacian Δ, and the square norm of the Ricci tensor $|r|^2$ are given as

$$s = 8, \quad \lambda_1 = 2, \quad |r|^2 = 14.$$

Let φ be the first eigenfunction of the Laplacian on the first component S^2. Then we have

$$\Delta\varphi = -2\varphi, \quad \int_M |d\varphi|^2 = 2\int_M \varphi^2$$

and thus, $E(\varphi) = 0$. Therefore, for $S^2 \times S^3$, we obtain

$$\mathcal{A}(\varphi) = 0 \quad \text{and} \quad \ker s_g'^* \neq 0.$$

For higher dimensional case, let $M = S^n \times S^{n+1}$ with the standard product metric. Then,

$$（2.4） \qquad s = 2n^2, \quad |r|^2 = n(2n^2 - n + 1)$$

and the first non-trivial eigenvalue is given as

$$\lambda_1(M) = \lambda_1(S^n) = n.$$

Let φ be the first eigenfunction corresponding to $\lambda_1(M)$ so that

$$（2.5） \qquad \Delta \varphi = -n\varphi, \quad r(d\varphi, d\varphi) = (n-1)|d\varphi|^2.$$

Substituting (2.4) and (2.5) into (2.2), we obtain $E(\varphi) = 0$. Therefore, we have $\mathcal{A}(\varphi) = 0$ and thus, $\ker s_g'^* \neq 0$.

Recall that $H^2(M) = W^{2,2}(M)$ is the Sobolev space consisting of functions that are L^2 up to the second (weak) derivative. Let

$$\mathcal{W} = \left\{ \varphi \in H^2(M) : \int_M \varphi^2 = 1 \right\}$$

and define

$$\nu = \inf \left\{ \int_M \varphi \mathcal{A}(\varphi) : \varphi \in \mathcal{W} \right\}.$$

Note that $\nu \geq 0$, and $\ker \mathcal{A} \neq 0$ implies $\nu = 0$ by (2.3). The converse is also true.

Theorem 2.5. *Suppose that $\nu = 0$. Then $\ker \mathcal{A} \neq 0$.*

Proof. Since $\nu = 0$, there exists a sequence (φ_k) of functions in $H^2(M)$ with $\int_M \varphi_k^2 = 1$ such that

$$E(\varphi_k) \to 0 \quad \text{as} \quad k \to \infty.$$

We now claim that (φ_k) is bounded in $H^2(M)$. On the contrary, suppose that the sequence (φ_k) is unbounded in $H^2(M)$. Defining $\widetilde{\varphi}_k$ as

$$\widetilde{\varphi}_k = \frac{\varphi_k}{\|\varphi\|_{H^2(M)}},$$

where $\|\varphi\|_{H^2(M)}$ denotes the Sobolev norm in $H^2(M)$, we have

$$\|\varphi\|_{H^2(M)} = 1 \quad \text{and} \quad \int_M \widetilde{\varphi}_k^2 \to 0 \text{ as } k \to \infty.$$

Furthermore, $E(\widetilde{\varphi}_k) \to 0$ as $k \to \infty$. Thus the rescaled sequence $(\widetilde{\varphi}_k)$ is bounded in $H^2(M)$ and so there exists a function $\widetilde{\varphi}_\infty \in H^2(M)$ such that $\widetilde{\varphi}_k \to \widetilde{\varphi}_\infty$ in $H^2(M)$.

However since $\|\widetilde{\varphi}_k\|_{L^2} \to 0$, the limit function $\widetilde{\varphi} = 0$, which is contradictory to the fact that $\|d\widetilde{\varphi}\|_{L^2} \neq 0$ or $\|Dd\widetilde{\varphi}\|_{L^2} \neq 0$.

Therefore, (φ_k) is bounded, and so φ_k weakly converges to a function φ in $H^2(M)$. Applying the Rellich-Kondrakov embedding theorem $H^2(M) \subset H^1(M) \subset L^2$, φ_k strongly converges to φ in L^2, and thus, there exists a subsequence, say (φ_k), that converges almost everywhere. Consequently, we have

$$E(\varphi) \leq \liminf_{k \to \infty} E(\varphi_k) = 0.$$

Hence since $E(\varphi) = 0$ and $\int_M \varphi^2 = 1$, φ is a non-constant function and $\mathcal{A}(\varphi) = 0$. $\qquad\square$

Corollary 2.6. $\nu = 0$ *if and only if* $\ker s_g'^* \neq 0$, *or if* $\ker \mathcal{A} \neq 0$.

Now we consider a special operator stemming from $\mathcal{A}$ that also plays a very important role in the kernel space of $s_g'^*$. For a function φ, define $P\varphi$ as

$$P\varphi = (n-1)\Delta^2\varphi + 2s_g\Delta\varphi - \langle Dd\varphi, r_g \rangle$$

and define

$$\mu = \inf_{\varphi \in H^2(M), \varphi \neq 0} \frac{\int \varphi P\varphi}{\int \varphi^2}.$$

Note that $\mu \leq 0$ since $P\varphi = 0$ when φ is a nonzero constant. Furthermore, it is easy to see that if $\mu = 0$, then either (M, g) is Ricci-flat or $\ker \mathcal{A} = 0$. In fact, if $u \in \ker \mathcal{A}$ and $r \neq 0$, then

$$\int_M uPu = -\int_M u^2|r|^2 \leq 0.$$

Since $\mu = 0$ and $r \neq 0$, u must be zero because $\int_M u^2|r|^2 = 0$. The following theorem shows that if $\ker \mathcal{A} \neq 0$, then μ must be strictly negative.

Theorem 2.7. *Assume that* $\ker \mathcal{A} \neq 0$ *and* $s = s_g$ *is constant. Then,*

$$-\max_M |r_g|^2 \leq \mu \leq -\frac{s_g^2}{n}.$$

Proof. Let $u \in \ker \mathcal{A}$ be a non-constant function and r be the Ricci tensor of the metric g. Since $\frac{s^2}{n} \leq |r|^2$, we have

$$\mu \int u^2 \leq \int uPu = -\int u^2|r|^2 \leq -\frac{s^2}{n} \int u^2.$$

Thus

$$\mu \leq -\frac{s^2}{n}.$$

On the other hand, it follows from Lemma 2.1, (2.1) and the Bochner-Weitzenböck formula that

$$\int_M (s_g'^* \varphi)^2 = \int_M \varphi \mathcal{A}(\varphi)$$
$$= \int_M \left\{ n(\Delta\varphi)^2 - 2s|d\varphi|^2 - |Dd\varphi|^2 + \varphi^2|r|^2 \right\}.$$

Thus,

$$\int_M \left\{ n(\Delta\varphi)^2 - 2s|d\varphi|^2 - |Dd\varphi|^2 \right\} \geq -\int_M \varphi^2|r|^2$$
$$\geq -\left(\max_M |r|^2 \right) \int_M \varphi^2.$$

Therefore, since

$$\int_M \varphi P \varphi \geq -\left(\max_M |r|^2 \right) \int_M \varphi^2$$

for any function φ, we conclude that

$$\mu \geq -\max_M |r|^2.$$

$\square$

In view of Theorem 2.7, the invariant μ may designate the criteria for how close g is to an Einstein metric. In fact, when (M, g) is Einsteinian, it follows from Theorem 2.7 that

$$\mu = -\frac{s^2}{n}$$

if $\ker \mathcal{A} \neq 0$.

In view of the operators $\mathcal{A}$ and P, for any real number α, we introduce an elliptic-fourth order partial differential equation $\mathcal{A}_\alpha$ defined as

$$\mathcal{A}_\alpha(\varphi) = (n-1)\Delta^2\varphi + 2s_g\Delta\varphi - \langle Dd\varphi, r_g \rangle + (1-\alpha)\varphi|r_g|^2,$$

where r_g is the Ricci tensor and s_g is the scalar curvature which is assumed to be a positive constant. Note that $\mathcal{A}_0 = \mathcal{A}$ and $\mathcal{A}_1 = P$.

Theorem 2.8. *Assume that* $\ker \mathcal{A} = 0$ *and* $s = s_g$ *is constant. Then there exists a positive real number* $\alpha_0 > 0$ *such that* $\ker \mathcal{A}_\alpha = 0$ *for all* α, $0 \leq \alpha \leq \alpha_0$.

Proof. For $0 < \alpha < 1$, let $u \in \ker \mathcal{A}_\alpha$ be a non-trivial function. Then

$$\mathcal{A}(u) = \alpha u |r_g|^2 \leq \left(\max_M |r|^2 \right) \alpha u$$

and so $\nu \leq (\max_M |r|^2) \alpha$. Since $\ker \mathcal{A} = 0$, Corollary 2.6 states that $\nu > 0$. Hence,

$$0 < \frac{\nu}{\max_M |r|^2} \leq \alpha.$$

$\square$

3. CASE OF $\nu > 0$

In this section, we consider the case in which ν is positive, or equivalently, $\ker \mathcal{A} = 0$. We will investigate some conditions and properties for ν to be positive, and derive lower bounds on ν.

Lemma 3.1. *Assume $\nu > 0$. We then obtain*

$$\inf_{\varphi \in \mathcal{W}, \varphi \neq 1} \frac{E(\varphi)}{\|\varphi\|_{H^2(M)}} > 0.$$

Here $\|\varphi\|_{H^2(M)}$ denotes the Sobolev norm in $H^2(M)$.

Proof. Suppose that $\inf_{\varphi \in \mathcal{W}, \varphi \neq 1} \dfrac{E(\varphi)}{\|\varphi\|_{H^2(M)}} = 0$. Then, there exists a sequence (φ_k) in $\mathcal{W}$ such that

$$\frac{E(\varphi_k)}{\|\varphi\|_{H^2(M)}} \to 0 \quad \text{as} \quad k \to \infty.$$

Since $\nu > 0$, we have $\|\varphi\|_{H^2(M)} \to \infty$ as $k \to \infty$. Defining $\widetilde{\varphi}_k$ as

$$\widetilde{\varphi}_k = \frac{\varphi_k}{\|\varphi\|_{H^2(M)}},$$

we can obtain a contradiction, as in the proof of Theorem 2.5. $\square$

Theorem 3.2. *Let (M, g) be a compact Riemannian n-manifold with a positive constant scalar curvature s. If $\ker \mathcal{A} = 0$, then $\nu > 0$ is contained in the spectrum of $\mathcal{A}$.*

Proof. Recall

$$\mathcal{W} = \left\{ \varphi \in H^2(M) : \int_M \varphi^2 = 1 \right\}.$$

By Theorem 2.5, we may assume that

$$a := \inf_{\varphi \in \mathcal{W}, \varphi \neq 1} \frac{E(\varphi)}{\|\varphi\|_{H^2(M)}} > 0.$$

Then, for any function $\varphi \in \mathcal{W}$, we have

$$E(\varphi) \geq a\|\varphi\|_{H^2(M)}$$

and thus,

$$E(\varphi) \to \infty \quad \text{as} \quad \|\varphi\|_{H^2(M)} \to \infty.$$

In other words, the functional E is coercive on $\mathcal{W}$.

On the other hand, let (φ_k) be a sequence in $H^2(M)$ such that $\varphi_k \to \varphi$ weakly in $H^2(M)$. Then, according to the Rellich-Kondrakov theorem, $\varphi_k \to \varphi$ strongly in $L^2(M)$ and thus, a subsequence (φ_k) converges almost everywhere. This shows that the subspace $\mathcal{W}$ is weakly closed in $H^2(M)$. Furthermore, since M is compact, the subsequence (φ_k) uniformly converges to φ, and we obtain

$$E(\varphi) \leq \liminf_{k \to \infty} E(\varphi_k).$$

It is a well-known fact that the functional E is bounded below and attains its minimum in $H^2(M)$ ([8]). Letting

$$E(u) = \min\{E(\varphi) : \varphi \in \mathcal{W}\},$$

it is easy to see from the variational principle that

$$\mathcal{A}(u) = \nu u.$$

$\square$

The properties of operator $\mathcal{A}$ and the lower bound on ν are deeply related to the first nonzero eigenvalue of the Laplacian. Let λ be the first nonzero eigenvalue for the Laplacian operator Δ, which is characterized by

$$\lambda = \inf\left\{ \frac{\int_M |\nabla\varphi|^2}{\int_M \varphi^2} \; : \; \int_M \varphi = 0 \right\}.$$

It is well-known that λ is positive. It follows from the characterization of the first nonzero eigenvalue that for any function φ with $\int_M \varphi = 0$,

$$(3.1) \qquad \int \varphi^2 \leq \frac{1}{\lambda} \int |d\varphi|^2.$$

Lemma 3.3. *Let (M^n, g) be a compact Riemannian n-manifold. Then, for any function $\varphi \in C^\infty(M)$,*

$$(3.2) \qquad \int_M |d\varphi|^2 \leq \frac{1}{\lambda} \int_M (\Delta\varphi)^2 \leq \frac{n}{\lambda} \int_M |Dd\varphi|^2,$$

where λ is the first nonzero eigenvalue of the Laplacian.

Proof. Applying the Cauchy-Schwarz inequality to φ and $\bar\varphi := \varphi - \int_M \varphi$, we can obtain

$$\int_M |d\varphi|^2 \le \frac{1}{\lambda} \int_M (\Delta\varphi)^2.$$

The second inequality follows from the fact that $(\Delta\varphi)^2 \le n|Dd\varphi|^2$. $\square$

Furthermore, for a function φ with $\int_M \varphi = 0$, we have

$$\int_M \varphi^2 \le \frac{1}{\lambda^2} \int_M (\Delta\varphi)^2 \quad \text{and} \quad \int_M \varphi^2 \le \frac{n}{\lambda^2} \int_M |Dd\varphi|^2.$$

A direct observation from the definition of $\mathcal{A}$ is the following. This theorem shows that if the first nonzero eigenvalue for the Laplacian is large compared to the sum of the scalar curvature and the norm of the Ricci tensor, then ν is positive.

Theorem 3.4. *Let (M^n, g) be a compact Riemannian n-manifold with a positive constant scalar curvature s. If $(n-1)\lambda \ge 2s + \max\limits_M |r|$, then $\nu \ge \frac{s^2}{n}$ and thus $\ker \mathcal{A} = 0$, or equivalently, $\ker s'^*_g = 0$.*

Proof. Note that $|r|^2 \ge \frac{s^2}{n}$. It follows from Lemma 3.3 that

$$\int_M |d\varphi|^2 \le \frac{1}{\lambda} \int_M (\Delta\varphi)^2$$

for any function φ. Thus for any function $\varphi \in \mathcal{W}$,

$$
\begin{aligned}
E(\varphi) &= \frac{1}{2} \int_M (n-1)(\Delta\varphi)^2 - 2s|d\varphi|^2 + r(d\varphi, d\varphi) + |r|^2\varphi^2 \\
&\ge \frac{1}{2} \{(n-1)\lambda - (2s + \max|r|)\} \int_M |d\varphi|^2 + \frac{s^2}{2n} \int_M \varphi^2.
\end{aligned}
$$

Hence, $\nu \ge \frac{s^2}{n}$. $\square$

Remark 3.5. Assume $\nu > 0$ for a compact Riemannian n-manifold (M, g) with a positive constant scalar curvature. Then it follows from Theorem 3.2 that

$$\mathcal{A}(u) = \nu u$$

for some function $u \in \mathcal{W}$. In particular, we have

$$\int_M u|r|^2 = \nu \int_M u.$$

Since $\ker s_g'^* = 0$, it is easy to see that the operator $\mathcal{A}$ is bijective ([?]). Thus there exists a unique function $\varphi \in C^\infty(M)$ such that $\mathcal{A}(\varphi) = u|r|^2$. Therefore

$$
\begin{aligned}
\frac{s^2}{n} &\leq \int_M u^2 |r|^2 = \int_M u\mathcal{A}\varphi \\
&= \int_M \varphi\mathcal{A}u = \nu \int_M \varphi u \\
&\leq \nu\|\varphi\|_{L^2}
\end{aligned}
$$

On the other hand, by the Cauchy-Schwarz inequality

$$
\begin{aligned}
\nu\|\varphi\|_{L^2}^2 &\leq \int_M \varphi\mathcal{A}\varphi = \int_M \varphi u|r|^2 \\
&\leq \left(\int_M \varphi^2 |r|^2\right)^{\frac{1}{2}} \left(\int_M u^2 |r|^2\right)^{\frac{1}{2}} \\
&\leq \left(\max_M |r|\right) \|\varphi\|_{L^2} \sqrt{\nu\|\varphi\|_{L^2}}.
\end{aligned}
$$

Therefore, we have $\nu\|\varphi\|_{L^2} \leq \max_M |r|^2$ and so

$$
\frac{s^2}{n} \leq \nu\|\varphi\|_{L^2} \leq \max_M |r|^2,
$$

where φ is a function satisfying $\mathcal{A}(\varphi) = u|r|^2$.

Theorem 3.6. *Let $M = S^m \times S^m$ $(m \geq 2)$ with the standard product metric. Then*

$$
\nu = m = \frac{\dim(M)}{2}.
$$

Proof. First, we will examine the case $m = 2$ since key ingredients of the proof are contained in this setting. The cases of $m \geq 3$ will then be briefly explained. For $M^4 = S^2 \times S^2$ with the standard product metric g, we obviously have $s = |r|^2 = 4, \lambda = 2$, and $r = g$. Thus, $\langle Dd\varphi, r\rangle = \Delta\varphi$ for any function φ, and so

$$
\mathcal{A}(\varphi) = 3\Delta^2\varphi + 7\Delta\varphi + 4\varphi.
$$

Let u be the first eigenfunction of S^2 so that $\Delta u = -2u, 2\int_M u^2 = \int_M |du|^2$ and $r(du, du) = |du|^2$. Therefore,

$$
\begin{aligned}
\int_M u\mathcal{A}(u) &= \int_M 3(\Delta u)^2 - 7|du|^2 + 4u^2 \\
&= 2\int_M u^2
\end{aligned}
$$

Hence $\nu \leq 2$. To show the converse inequality $\nu \geq 2$, it is sufficient to prove that for any C^∞ function φ

$$F(\varphi) := \int_M \left[3(\Delta\varphi)^2 - 7|d\varphi|^2 + 2\varphi^2\right] \geq 0.$$

First, note that

$$F(\varphi) = \int_M (\Delta\varphi + 2\varphi)(3\Delta\varphi + \varphi).$$

It follows from Lemma 3.3 that

$$2\int_M |d\varphi|^2 \leq \int_M (\Delta\varphi)^2.$$

Thus, from the monotonicity of eigenvalues, it follows that, for any function φ that vanishes on the smooth boundary ∂D of a domain $D \subset M$, we have

$$(3.3) \qquad 2\int_D |d\varphi|^2 \leq \int_D (\Delta\varphi)^2.$$

Assume for a moment that 0 is a regular value of φ. Let D_1 be a region on M such that

$$\Delta\varphi + 2\varphi \leq 0 \quad \text{and} \quad \Delta\varphi + \varphi \geq 0,$$

and D_2 be a region such that

$$\Delta\varphi + 2\varphi \geq 0 \quad \text{and} \quad \Delta\varphi + \varphi \leq 0.$$

Note that $\varphi \leq 0$ on region D_1, and $\varphi \geq 0$ on region D_2. Thus, $\partial D_1 = \partial D_2$. On region D_1, we have

$$(3.4) \qquad 0 < -\frac{1}{3}\varphi \leq \Delta\varphi \leq -2\varphi.$$

Multiplying (3.4) by φ and integrating over D_1, we obtain

$$-2\int_{D_1} \varphi^2 \leq \int_{D_1} \varphi\Delta\varphi \leq -\frac{1}{3}\int_{D_1} \varphi^2.$$

Since $\varphi = 0$ on ∂D_1, we get

$$(3.5) \qquad -2\int_{D_1} \varphi^2 \leq -\int_{D_1} |d\varphi|^2 \leq -\frac{1}{3}\int_{D_1} \varphi^2.$$

Similarly, on region D_2, we have

$$(3.6) \qquad -2\int_{D_2} \varphi^2 \leq \int_{D_2} \varphi\Delta\varphi \leq -\frac{1}{3}\int_{D_2} \varphi^2.$$

Let $D = D_1 \cup D_2$. It follows from (3.5) and (3.6) that

$$(3.7) \qquad \frac{1}{3}\int_D \varphi^2 \leq \int_D |d\varphi|^2 \leq 2\int_D \varphi^2.$$

Note that on $M - D$, we have

$$(3.8) \qquad (\Delta\varphi + 2\varphi)(3\Delta\varphi + \varphi) \geq 0.$$

Furthermore, since the function φ vanishes on the boundary ∂D of D, we can apply integration by parts and Green's identity. Thus, it follows from (3.3), (3.7), and (3.8) that

$$
\begin{aligned}
F(\varphi) &= \int_D (\Delta\varphi + 2\varphi)(3\Delta\varphi + \varphi) + \int_{M-D} (\Delta\varphi + 2\varphi)(3\Delta\varphi + \varphi) \\
&= 3\int_D \left[(\Delta\varphi)^2 - 2|d\varphi|^2 \right] + \int_D \left(2\varphi^2 - |d\varphi|^2 \right) \\
&\quad + \int_{M-D} (\Delta\varphi + 2\varphi)(3\Delta\varphi + \varphi) \\
&\geq 0.
\end{aligned}
$$

Now, assume that 0 is a critical value of φ. By Sard's theorem, for any positive real number $\epsilon > 0$, there exists a real number $a, -\epsilon < a < 0$ such that a is a regular value φ. Let $D_{1,a}$ be a region such that

$$(3.9) \qquad \Delta\varphi + 2\varphi \leq \frac{5}{3}a \quad \text{and} \quad \Delta\varphi + \frac{\varphi}{3} \geq 0.$$

Note that $\varphi \leq a < 0$ on region $D_{1,a}$, and $\varphi = a$ on the boundary $\partial D_{1,a}$. Multiplying (3.9) by φ and integrating it over $D_{1,a}$, we obtain

$$(3.10) \qquad \frac{5}{3}a \int_{D_{1,a}} \varphi - a \int_{\partial D_{1,a}} \frac{\partial\varphi}{\partial n_1} \leq \int_{D_{1,a}} \left(2\varphi^2 - |d\varphi|^2 \right),$$

where n_1 is a unit normal vector field to $\partial D_{1,a}$. Next, let $D_{2,a}$ be a region such that

$$(3.11) \qquad \Delta\varphi + 2\varphi \geq -\frac{5}{3}a \quad \text{and} \quad \Delta\varphi + \frac{\varphi}{3} \leq 0.$$

We may assume that $-a$ is also a regular value of φ. Note that $0 < -a \leq \varphi$ on region $D_{2,a}$, and $\varphi = -a$ on the boundary $\partial D_{2,a}$. Multiplying (3.11) by φ and integrating it over $D_{2,a}$, we obtain

$$(3.12) \qquad a \int_{\partial D_{2,a}} \frac{\partial\varphi}{\partial n_2} - \frac{5}{3}a \int_{D_{2,a}} \varphi \leq \int_{D_{2,a}} \left(2\varphi^2 - |d\varphi|^2 \right),$$

where n_2 is a unit normal vector field on $\partial D_{2,a}$. Decomposing M into three regions, we can write

$$
\begin{aligned}
F(\varphi) \;=\; & 3 \int_{D_{1,a}} \left[(\Delta\varphi)^2 - 2|d\varphi|^2 \right] + \int_{D_{1,a}} \left(2\varphi^2 - |d\varphi|^2 \right) \\
& + 3 \int_{D_{2,a}} \left[(\Delta\varphi)^2 - 2|d\varphi|^2 \right] + \int_{D_{2,a}} \left(2\varphi^2 - |d\varphi|^2 \right) \\
& + \int_{M-(D_{1,a} \cup D_{2,a})} (\Delta\varphi + 2\varphi)(3\Delta\varphi + \varphi).
\end{aligned}
$$

Applying inequality (3.3) to $\varphi - a$, we have

$$
\int_{D_{1,a}} \left[(\Delta\varphi)^2 - 2|d\varphi|^2 \right] \geq 0
$$

and

$$
\int_{D_{2,a}} \left[(\Delta\varphi)^2 - 2|d\varphi|^2 \right] \geq 0.
$$

Thus, from (3.10) and (3.12), we obtain

$$
\begin{aligned}
F(\varphi) \;\geq\; & \frac{5}{3}|a| \int_{D_{1,a} \cup D_{2,a}} |\varphi| - a \int_{\partial D_{1,a}} \frac{\partial \varphi}{\partial n_1} + a \int_{\partial D_{2,a}} \frac{\partial \varphi}{\partial n_2} \\
& + \int_{M-(D_{1,a} \cup D_{2,a})} (\Delta\varphi + 2\varphi)(3\Delta\varphi + \varphi).
\end{aligned}
$$

Since $|\frac{\partial \varphi}{\partial n_1}| \leq |d\varphi|$ and $|\frac{\partial \varphi}{\partial n_2}| \leq |d\varphi|$, the first three terms on the right-hand side approach to 0 as $\epsilon \to 0$. Finally, let $E_{1,a}$ be a region such that $\Delta\varphi + 2\varphi > \frac{5}{3}a$ and $\Delta\varphi + \frac{\varphi}{3} \geq 0$, and $E_{2,a}$ be a region such that $\Delta\varphi + 2\varphi < -\frac{5}{3}a$ and $\Delta\varphi + \frac{\varphi}{3} \leq 0$. Then, we have

$$
\begin{aligned}
\int_{M-(D_{1,a} \cup D_{2,a})} & (\Delta\varphi + 2\varphi)(3\Delta\varphi + \varphi) \\
& \geq \frac{5}{3}a \int_{E_{1,a}} (3\Delta\varphi + \varphi) - \frac{5}{3}a \int_{E_{2,a}} (3\Delta\varphi + \varphi).
\end{aligned}
$$

The right-hand side approaches to 0 as $\epsilon \to 0$. Hence, $F(\varphi) \geq 0$.

In the general case, $M^{2m} = S^m \times S^m$ when $m \geq 2$, it is easy to see that

$$
s = 2m(m-1), \quad |r|^2 = 2m(m-1)^2, \quad r = (m-1)g, \quad \lambda = m.
$$

Thus,

$$
\begin{aligned}
\int_M \varphi \mathcal{A}(\varphi) \;=\; & (2m-1) \int_M \left[(\Delta\varphi)^2 - m|d\varphi|^2 \right] \\
& - (2m^2 - 4m + 1) \int_M |d\varphi|^2 + 2m(m-1)^2 \int_M \varphi^2.
\end{aligned}
$$

Using the first eigenfunction u of S^m, $\Delta u = -mu$, we can demonstrate that $\nu \leq m$. To show that $\nu \geq m$, it is sufficient to prove that for any function φ

$$F(\varphi) := \int_M (\Delta\varphi + m\varphi) \left[(2m-1)\Delta\varphi + (2m^2 - 4m + 1)\varphi \right] \geq 0.$$

Note that

$$m \int_M |d\varphi|^2 \leq \int_M (\Delta\varphi)^2.$$

An argument identical to that used in the case $S^2 \times S^2$ shows that $F(\varphi) \geq 0$, and thus, $\nu = m$. $\qquad\square$

Remark 3.7. For the case of $M = S^m \times S^{m+k}$ with $k \geq 2$, the first nonzero eigenfunction of S^m can be used to show that

$$\nu \leq \min\{ (m+k)(k-1)^2, \ m(k+1)^2 \}.$$

However, we do not know the exact lower bound on ν.

4. The First Eigenvalue of the Laplacian

As mentioned above, the first non-zero eigenvalue $\lambda = \lambda_1(M)$ of the Laplace operator for a Riemannian manifold (M, g) is related to the operator $\mathcal{A}$. For example, if $\ker \mathcal{A} \neq 0$ and g is an Einstein metric with a positive scalar curvature, then $\lambda = \frac{s}{n-1}$ from the results obtained by Lichnerowicz ([2]) and Bourguignon ([4]). We shall now see that, if there is a non-trivial function on which the action of $\mathcal{A}$ is non-positive where the function is positive, then the first nonzero eigenvalue of the Laplacian is bounded above and vice versa. Recall that we assumed that the scalar curvature $s_g = s$ of a Riemannian manifold (M, g) is always a positive constant.

For a function φ on a smooth manifold M, let us denote

$$M_\varphi^+ = \{x \in M \ : \ \varphi(x) > 0\}.$$

We say that a Riemannian manifold (M, g) satisfies the $\mathcal{A}$-*superharmonic condition* if there exists a smooth function φ such that

(i) $M_\varphi^+ \neq \emptyset$ and $\mathcal{A}\varphi \leq 0$ on M_φ^+,
(ii) $\Delta\varphi = 0$ on the boundary ∂M_φ^+ of M_φ^+.

For example, if $M = S^n$ with the standard round metric g_0, and φ is the first nonzero eigenfunction of the Laplacian, i.e., $\Delta\varphi = -n\varphi$, then $\mathcal{A}\varphi = 0$ and (S^n, g_0) satisfies

the $\mathcal{A}$-superharmonic condition. Furthermore, note that any eigenfunction of the Laplacian satisfies the second condition (ii). The following lemma shows that the $\mathcal{A}$-superharmonic condition is implied by $\ker \mathcal{A} \neq 0$.

Lemma 4.1. *Let (M^n, g) be a compact n-dimensional Riemannian manifold with a positive constant scalar curvature s_g. If $\ker \mathcal{A} \neq 0$, then (M, g) satisfies the $\mathcal{A}$-superharmonic condition.*

Proof. By Lemma 2.3, $\ker \mathcal{A} \neq 0$ is equivalent to $\ker s_g'^{*} \neq 0$. Let $s_g'^{*}\varphi = 0$ and $\varphi \neq 0$. Then,

$$Dd\varphi - (\Delta\varphi)g - \varphi r_g = 0.$$

In particular, taking the trace yields

$$\Delta\varphi = -\frac{s_g}{n-1}\varphi$$

and so $M_\varphi^+ \neq \emptyset$. On the other hand, taking the inner product with the Ricci tensor r_g of g, we obtain

$$\langle Dd\varphi, r_g \rangle - s_g\Delta\varphi - \varphi|r_g|^2 = 0.$$

Thus, it follows from the definition of $\mathcal{A}$ that

$$
\begin{aligned}
\mathcal{A}\varphi &= (n-1)\Delta^2\varphi + 2s_g\Delta\varphi - \langle Dd\varphi, r_g \rangle + \varphi|r_g|^2 \\
&= \frac{s_g^2}{n-1}\varphi - \frac{s_g^2}{n-1}\varphi \\
&= 0.
\end{aligned}
$$

Hence, the function φ satisfies conditions (i) and (ii) in the definition of the $\mathcal{A}$-superharmonic condition. $\qquad\square$

Theorem 4.2. *Let (M^n, g) be a compact n-dimensional Riemannian manifold with a positive constant scalar curvature s_g. Suppose that (M, g) satisfies the $\mathcal{A}$-superharmonic condition. If $r_g \geq k$ for $0 < k \leq 2\left(1 - \sqrt{1 - \frac{1}{n}}\right)s_g$, then the first eigenvalue of the Laplacian satisfies*

$$(4.1) \qquad \lambda = \lambda_1(M) \leq \frac{2s_g - k + \sqrt{k^2 - 4ks_g + \frac{4s_g^2}{n}}}{2(n-1)}.$$

Proof. Let $s_g = s$ and $r_g = r$. In addition, let φ be a smooth function satisfying $M_\varphi^+ \neq \emptyset$, $\mathcal{A}\varphi \leq 0$ on M_φ^+, and $\Delta\varphi = 0$ on the boundary ∂M_φ^+. If φ is a constant

function, then φ is a positive constant since $M_\varphi^+ \neq \emptyset$. However, we have $\mathcal{A}\varphi = \varphi|r|^2 \leq 0$, which is a contradiction. Thus, we may assume that φ is a non-constant function. By the above hypothesis, we have

$$(4.2) \qquad \int_{M_\varphi^+} \varphi \mathcal{A}\varphi \leq 0.$$

By the definition of $\mathcal{A}$ and integration by parts, together with the fact that $\Delta\varphi = 0$ on ∂M_φ^+, we obtain

$$\int_{M_\varphi^+} \varphi \mathcal{A}\varphi = \int_{M_\varphi^+} (n-1)(\Delta\varphi)^2 - \int_{\partial M_\varphi^+} \Delta\varphi \frac{\partial\varphi}{\partial\nu}$$

$$+ \int_{M_\varphi^+} \left[2s\varphi\Delta\varphi + r(d\varphi, d\varphi) + |r|^2\varphi^2 \right]$$

$$(4.3) \qquad \geq \int_{M_\varphi^+} \left[(n-1)(\Delta\varphi)^2 + (2s-k)\varphi\Delta\varphi + \frac{s^2}{n}\varphi^2 \right].$$

Note that

$$(4.4) \qquad (n-1)(\Delta\varphi)^2 + (2s-k)\varphi\Delta\varphi + \frac{s^2}{n}\varphi^2$$

$$= ((n-1)\Delta\varphi + \alpha\varphi)(\Delta\varphi + \beta\varphi),$$

where

$$\alpha = \frac{2s - k + \sqrt{k^2 - 4ks + \frac{4s^2}{n}}}{2}, \quad \beta = \frac{2s - k - \sqrt{k^2 - 4ks + \frac{4s^2}{n}}}{2(n-1)}.$$

Also note that

$$k = 2s\left(1 - \sqrt{1 - \frac{1}{n}}\right) \quad \Rightarrow \quad k^2 - 4ks + \frac{4s^2}{n} = 0.$$

Contention: If

$$(4.5) \qquad \lambda = \lambda_1(M) > \frac{2s - k + \sqrt{k^2 - 4ks + \frac{4s^2}{n}}}{2(n-1)} = \frac{\alpha}{n-1},$$

then any subset Ω of M_φ^+ with a C^1 boundary on which $(n-1)\Delta\varphi + \alpha\varphi \geq 0$ and $\Delta\varphi + \beta\varphi \leq 0$ has a measure of zero.

Suppose that a subset Ω of M_φ^+ contains an open n-ball. Note that since $\Delta\varphi = \varphi = 0$ on $\partial\Omega$, we can apply the Dirichlet principle on the first nonzero eigenvalue of the Laplacian. By monotonicity, we have

$$\lambda \leq \lambda_1(\Omega).$$

Since $(n-1)\Delta\varphi + \alpha\varphi \geq 0$ and $\varphi > 0$ on Ω, we have

$$\varphi\Delta\varphi \geq -\frac{\alpha}{n-1}\varphi^2.$$

Integrating this over Ω, we obtain

$$\int_\Omega |d\varphi|^2 \leq \frac{\alpha}{n-1}\int_\Omega \varphi^2 \leq \frac{\alpha}{n-1}\cdot\frac{1}{\lambda_1(\Omega)}\int_\Omega |d\varphi|^2.$$

Thus,

$$1 \leq \frac{\alpha}{n-1}\cdot\frac{1}{\lambda_1(\Omega)}$$

and so

$$\lambda \leq \lambda_1(\Omega) \leq \frac{\alpha}{n-1},$$

which contradicts (4.5). This completes the proof of contention.

Now, suppose that $\lambda > \frac{\alpha}{n-1}$. Since $\alpha > (n-1)\beta$, it follows from (4.4) and the above contention that

$$(n-1)(\Delta\varphi)^2 + (2s-k)\varphi\Delta\varphi + \frac{s^2}{n}\varphi^2 \geq 0 \quad \text{a.e on } M_\varphi^+,$$

which implies that $\int_{M_\varphi^+} \varphi\mathcal{A}\varphi \geq 0$. Consequently, from (4.2), we have

$$\int_{M_\varphi^+} \varphi\mathcal{A}\varphi = 0.$$

Thus, on the set M_φ^+, we have $\mathcal{A}\varphi = 0$ and

$$(n-1)(\Delta\varphi)^2 + (2s-k)\varphi\Delta\varphi + \frac{s^2}{n}\varphi^2$$
$$= ((n-1)\Delta\varphi + \alpha\varphi)(\Delta\varphi + \beta\varphi)$$
$$= 0$$

by (4.3). Since $\alpha > (n-1)\beta$, either $(n-1)\Delta\varphi + \alpha\varphi = 0$ or $\Delta\varphi + \beta\varphi = 0$ on the entire set M_φ^+. Therefore, we obtain

$$\lambda \leq \lambda_1(M_\varphi^+) \leq \max\left\{\frac{\alpha}{n-1}, \beta\right\} = \frac{\alpha}{n-1},$$

which contradicts the assumption $\lambda > \frac{\alpha}{n-1}$. Hence,

$$\lambda = \lambda_1(M) \leq \frac{\alpha}{n-1}.$$

$\square$

Remark 4.3. From the work of Lichnerowicz ([2]), if a compact Riemannian manifold (M^n, g) satisfies $r_g \geq k > 0$, then

$$\lambda = \lambda_1(M) \geq \frac{n}{n-1} k.$$

On the other hand, if $M = S^n$ with the standard round metric,

$$n - 1 \leq 2 \left(1 - \sqrt{1 - \frac{1}{n}} \right) s_g.$$

Thus, taking $k = n - 1$, the right-hand side in inequality (4.1) becomes

$$\frac{2s_g - k + \sqrt{k^2 - 4ks_g + \frac{4s_g^2}{n}}}{2(n-1)} = n$$

and so the result in Theorem 4.2 is optimal. Also, note that if $k = 2s \left(1 - \sqrt{1 - \frac{1}{n}} \right)$, then $k^2 - 4ks_g + \frac{4s_g^2}{n} = 0$ and thus, the following fact holds: if (M, g) satisfies the $\mathcal{A}$-superharmonic condition and $r_g \geq 2s \left(1 - \sqrt{1 - \frac{1}{n}} \right)$, then

$$\lambda_1(M) \leq \frac{s}{2\sqrt{n(n-1)}}.$$

Without the condition on k in Theorem 4.2 and with only the nonnegativity of the Ricci curvature, we obtain the following slightly weaker result. In fact, this is the $k = 0$ case in Theorem 4.2.

Corollary 4.4. *Let (M^n, g) be a compact n-dimensional Riemannian manifold such that the Ricci curvature is nonnegative and the scalar curvature s_g is a positive constant. In addition, suppose that (M, g) satisfies the $\mathcal{A}$-superharmonic condition. Then, the first nonzero eigenvalue of the Laplacian satisfies*

$$\lambda = \lambda_1(M) \leq \frac{s_g}{n-1} \left(1 + \frac{1}{\sqrt{n}} \right).$$

Proof. Let $s_g = s$, $\mathcal{A}\varphi \leq 0$ on $M_\varphi^+ \neq \emptyset$, and $\Delta\varphi = 0$ on ∂M_φ^+ for a function φ. Then, we have

$$
\begin{aligned}
(4.6) \qquad 0 &\geq \int_{M_\varphi^+} \varphi \mathcal{A}\varphi \\
&= \int_{M_\varphi^+} (n-1)(\Delta\varphi)^2 + 2s\varphi\Delta\varphi + r(d\varphi, d\varphi) + |r|^2\varphi^2 \\
&\geq \int_{M_\varphi^+} \left[(n-1)(\Delta\varphi)^2 + 2s\varphi\Delta\varphi + \frac{s^2}{n}\varphi^2 \right] + \int_{M_\varphi^+} r(d\varphi, d\varphi) \\
&=: I_1 + I_2.
\end{aligned}
$$

Note that

$$(n-1)(\Delta\varphi)^2 + 2s\varphi\Delta\varphi + \frac{s^2}{n}\varphi^2 = ((n-1)\Delta\varphi + a\varphi)(\Delta\varphi + b\varphi),$$

where

$$a = \left(1 + \frac{1}{\sqrt{n}}\right)s, \quad b = \frac{1}{n-1}\left(1 - \frac{1}{\sqrt{n}}\right)s.$$

As in the proof of Theorem 4.2, we can show that if

$$\lambda = \lambda_1(M) > \frac{s}{n-1}\left(1 + \frac{1}{\sqrt{n}}\right) = \frac{a}{n-1},$$

then any subset Ω with a C^1 boundary of M_φ^+ on which $(n-1)\Delta\varphi + \alpha\varphi \geq 0$ and $\Delta\varphi + \beta\varphi \leq 0$ has a measure of zero. Since $r_g \geq 0$, this implies that

$$(n-1)(\Delta\varphi)^2 + 2s\varphi\Delta\varphi + \frac{s^2}{n}\varphi^2 = 0 \quad \text{a.e on } M_\varphi^+.$$

Since $a > (n-1)b$, either $(n-1)\Delta\varphi + a\varphi = 0$ or $\Delta\varphi + b\varphi = 0$ on the entire set M_φ^+. This contradicts the assumption $\lambda_1(M) > \frac{a}{n-1}$ from the monotonicity of the eigenvalues of the Laplacian. $\square$

Remark 4.5. *Let (M^n, g) be a compact n-dimensional Riemannian manifold such that $r_g \geq k$ for $0 \leq k \leq 2\left(1 - \sqrt{1 - \frac{1}{n}}\right)s_g$, where the scalar curvature s_g is a positive constant. In addirion, suppose that there exists a function φ such that $M_\varphi^- = \{x \in M : \varphi(x) < 0\} \neq \emptyset$ and $\mathcal{A}\varphi \geq 0$ on M_φ^-. Then, the same argument used in the proof of Theorem 4.2 shows that the first nonzero eigenvalue satisfies*

$$\lambda = \lambda_1(M) \leq \frac{2s_g - k + \sqrt{k^2 - 4ks_g + \frac{4s_g^2}{n}}}{2(n-1)}.$$

In particular, if $k = 0$, then

$$\lambda = \lambda_1(M) \leq \frac{s}{n-1}\left(1 + \frac{1}{\sqrt{n}}\right).$$

Finally, we consider the relationship of ν with the first nonzero eigenvalue of the Laplace operator. In the case of $\nu > 0$, it follows from Theorem 3.2 that a minimizer u for the functional E satisfies $\mathcal{A}u = \nu u$. In particular, since $\ker s_g'^* = 0$ when $\nu > 0$, we cannot, in general, expect that $\frac{s_g}{n-1}$ is contained in the spectrum of the Laplace operator.

Theorem 4.6. *Let (M, g) be a compact n-dimensional Riemannian manifold such that $r_g \geq k \geq 0$ and assume that $\nu > \frac{s_g^2}{n}$, where s_g is a positive constant. In addition,*

suppose that $M_u^+ \neq \emptyset$ for a function u satisfying $\mathcal{A}u = \nu u$. Then, the first nonzero eigenvalue of the Laplacian satisfies

$$\lambda_1(M) \leq \frac{2s_g - k + \sqrt{k^2 - 4ks_g + \frac{4s_g^2}{n} + 4(n-1)\nu}}{2(n-1)},$$

unless (M, g) is Einsteinian.

Proof. We shall denote s_g as s and r_g as r. From $\int_M u\mathcal{A}u = \nu \int_M u^2$,

$$\begin{aligned}
0 &= \int_M u\mathcal{A}u - \nu u^2 \\
&= \int_M (n-1)(\Delta u)^2 + 2su\Delta u + r(du, du) + (|r|^2 - \nu)u^2 \\
&\geq \int_M (n-1)(\Delta u)^2 + (2s-k)u\Delta u + \left(\frac{s^2}{n} - \nu\right)u^2
\end{aligned}$$

We may factor the integrand as follows:

$$(n-1)(\Delta u)^2 + (2s-k)u\Delta u + \left(\frac{s^2}{n} - \nu\right)u^2$$
$$= ((n-1)\Delta u + \alpha u)\left(\Delta u + \frac{\beta}{n-1}u\right),$$

where

$$\begin{aligned}
\alpha &= \frac{1}{2}\left(2s - k + \sqrt{k^2 - 4ks + \frac{4s^2}{n} + 4(n-1)\nu}\right), \\
\beta &= \frac{1}{2}\left(2s - k - \sqrt{k^2 - 4ks + \frac{4s^2}{n} + 4(n-1)\nu}\right).
\end{aligned}$$

Note that if $\nu > \frac{s^2}{n}$, $k^2 - 4ks + \frac{4s^2}{n} + 4(n-1)\nu > 0$ for any $k \geq 0$.

The remainder of the proof is similar to that of Theorem 4.2. Hence, if g is not an Einstein memtric and $\lambda > \frac{\alpha}{n-1}$, then

$$0 \geq \int_M u\mathcal{A}u - \nu u^2 > 0,$$

which is a contradiction. $\square$

Corollary 4.7. *Let (M, g) be a compact n-dimensional Riemannian manifold of positive Ricci curvature. In addition, assume that $M_u^+ \neq \emptyset$ for a function u satisfying $\mathcal{A}u = \nu u$. Then, the first nonzero eigenvalue of the Laplacian satisfies*

$$\lambda_1 \leq \frac{s_g + \sqrt{\frac{s_g^2}{n} + (n-1)\nu}}{(n-1)}.$$

Theorem 4.8. *Let (M, g) be a compact n-dimensional Riemannian manifold such that $r_g \geq k$ with $0 \leq k \leq 2s_g \left(1 - \sqrt{1 - \frac{1}{n} - (n-1)\frac{\nu}{s_g^2}}\right)$. Suppose that $0 < \nu \leq \frac{s_g^2}{n}$. In addition, assume that $M_u^+ \neq \emptyset$ for a function u satisfying $\mathcal{A}u = \nu u$. Then, the first nonzero eigenvalue of the Laplacian satisfies*

$$\lambda_1 \leq \frac{2s_g - k + \sqrt{k^2 - 4ks_g + \frac{4s_g^2}{n} + 4(n-1)\nu}}{2(n-1)},$$

unless (M, g) is Einstein.

Proof. Note that if $\nu \leq \dfrac{s_g^2}{n}$ and $0 \leq k \leq 2s_g \left(1 - \sqrt{1 - \frac{1}{n} - (n-1)\frac{\nu}{s_g^2}}\right)$, then

$$k^2 - 4ks_g + \frac{4s_g^2}{n} + 4(n-1)\nu \geq 0.$$

The remainder of the proof proceeds in the same manner as Theorem 4.6. $\qquad\square$

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